Analysis 1

26 March 2024

Types of functions

Last time

Assume a, b, c, d, n are constants.

- $f(x) = ax^n + bx^{n-1} + \cdots + cx + d$ is a **polynomial**.
- $f(x) = \dfrac{\text{one polynomial}}{\text{another polynomial}}$ is a **rational function**.
- $f(x) = a\,x^b$ is a **power function**. This includes $5x^{1/2}$, which is $5\sqrt{x}$.
- $f(x) = a\,b^x$ is an **exponential function**.
- $f(x) = a\sin(bx + c) + d$ is a **trigonometric function** (... $\cos$ ... is too).
- $f(x) = a\ln(bx + c) + d$ is a **logarithmic function**.

A **piecewise function** uses different formulas for different inputs.

Combining functions

Last time

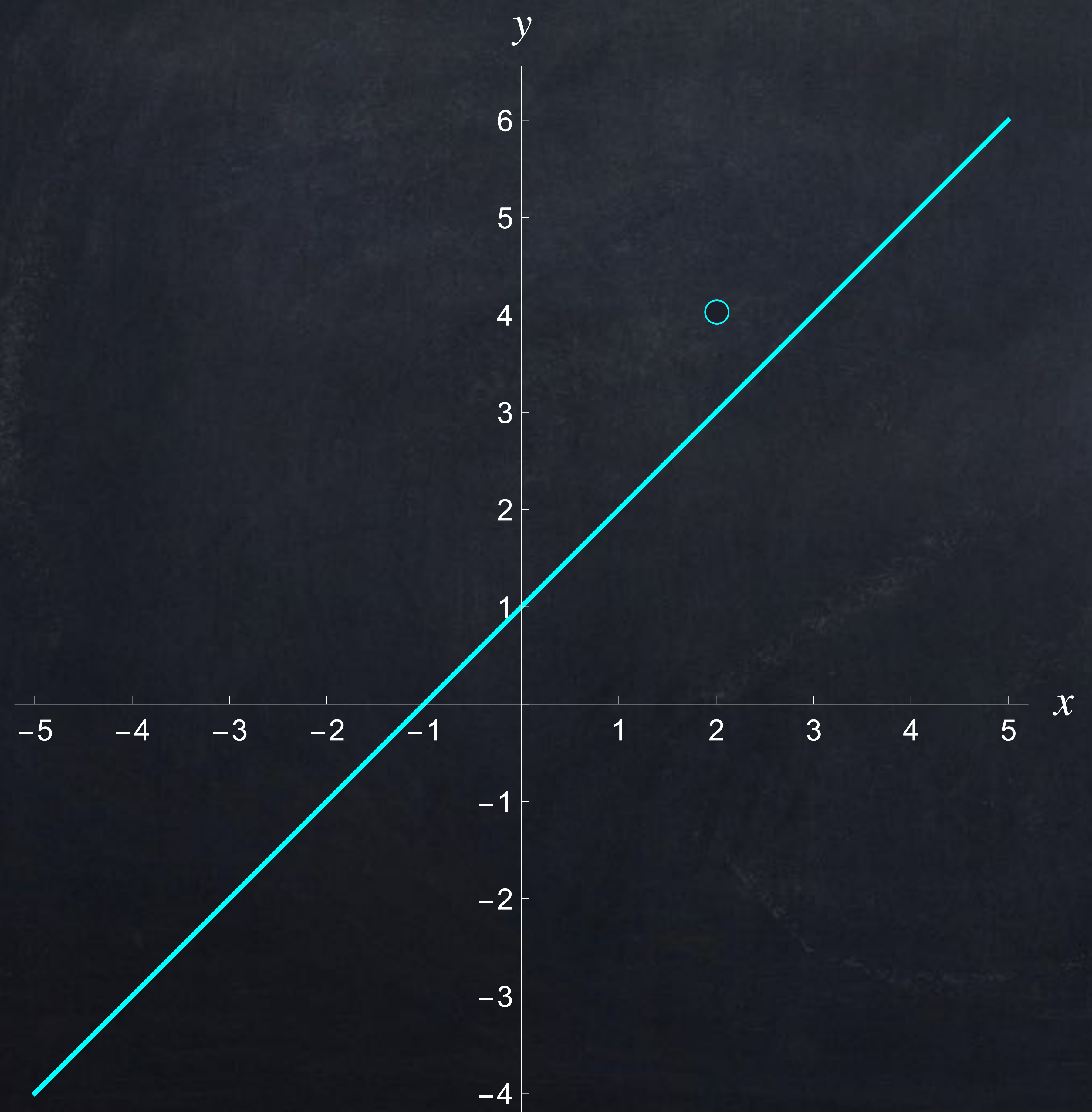
Given two functions $f(x)$ and $g(x)$, we can create the

- **sum** $f(x) + g(x)$,
- **difference** $f(x) - g(x)$,
- **product** $f(x) \cdot g(x)$,
- **quotient** $\dfrac{f(x)}{g(x)}$, and
- **composition** $f(g(x))$.

The first four words are also used for numbers (e.g., 5 is the sum of 2 and 3), but composition is only used for functions.

Limits

For the function $f(x) = \dfrac{x^2 - x - 2}{x - 2}$,



All of the x-values very close to 2 give us values of $f(x)$ very close to 3.

In symbols, we write

$$\lim_{x \to 2} f(x) = 3$$

for this function.

Ideas vs. Calculations vs. Applications

Mathematics can be interesting to study on its own, but of course it is also very useful, especially in science and engineering.

- *<u>What</u>* does $\lim_{x \to \infty} \frac{x^2}{3x + \sqrt{x}}$ *mean*?
- *<u>How</u>* do you *calculate* $\lim_{x \to \infty} \frac{x^2}{3x + \sqrt{x}}$?
- *<u>Why</u>* would this be *helpful* to calculate?

  e.g., comparing algorithm complexity (Comp. Sci.)

Ideas vs. Calculations vs. Applications

Mathematics can be interesting to study on its own, but of course it is also very useful, especially in science and engineering.

- *What* does $\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{x^2}{3x+\sqrt{x}}\right]$ *mean*?
- *How* do you *calculate* $\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{x^2}{3x+\sqrt{x}}\right]$?
- *Why* would this be *helpful* to calculate?

Usually Analysis 1 spends several weeks on limits, but this semester we will work on calculating derivatives first.

Derivative

The **derivative** of the function $f(x)$ is a new function that we can write as

$$f'(x) \quad \text{or} \quad f' \quad \text{or} \quad \frac{\mathrm{d}f}{\mathrm{d}x} \quad \text{or} \quad D[f].$$

Its official definition uses limits.

For now we will focus on specific patterns of functions whose derivatives we can calculate.

- Example: if $f(x) = \frac{1}{4}x^2$ then $f'(x) = \frac{1}{2}x$.

Often we are interested in the derivative at a specific value of x.

- Example, if $f(x) = \frac{1}{4}x^2$ then $f'(3) = \frac{3}{2}$.

Derivative

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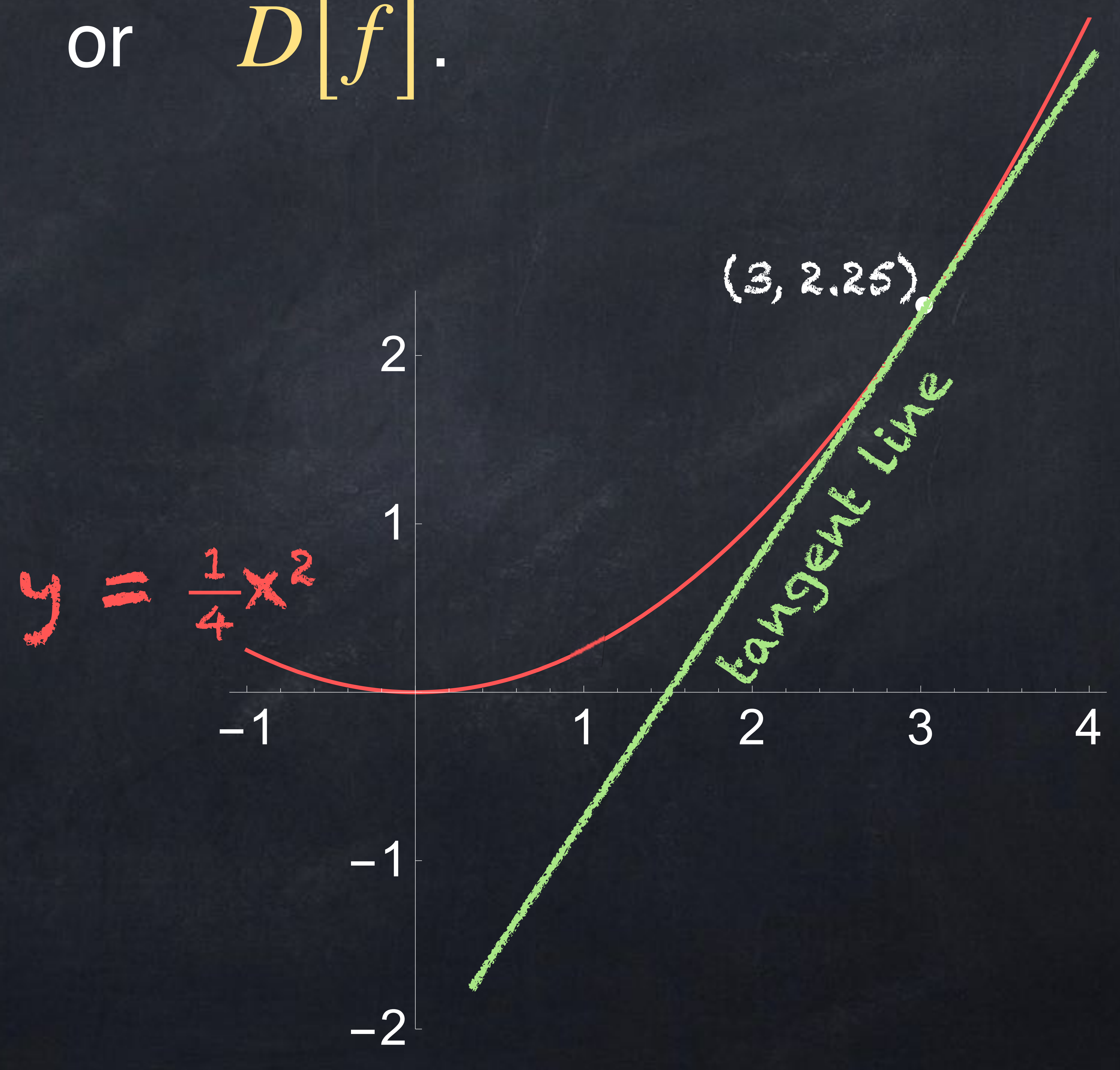
Its official definition uses limits.

One common description is that $f'(3)$ is the slope of the tangent line to the graph $y = f(x)$ at the point $(3, f(3))$.

On the right we can see that a tangent line to $y = \frac{1}{4}x^2$ at $(3, \frac{9}{4})$ does have slope $\frac{3}{2}$.



Derivative

- No matter what f represents, f' is a rate of change of f.
- Slope is the rate of change of y-position with respect to x-position.
- Velocity is the rate of change of position with respect to time.
- Acceleration is the rate of change of velocity with respect to time.
- Power is the rate of change of energy with respect to time.
- Current is the rate of change of charge with respect to time.
- Force is the rate of change of work with respect to position.
- Force is the rate of change of momentum with respect to time.
- Electric field is the rate of change of –voltage with respect to position.

Derivative

Your other classes will show you some ways derivatives can be used.

We will also cover some later (e.g., finding local maximum of a function).

For today, we will deal with calculations:

- If $f(x) = 3x^5$, then $f'(x) = ?$.
  - Same question: $(3x^5)' = ?$.
  - Same question: $D\left[3x^5\right] = ?$.
  - Same: Find the derivative of $3x^5$.
  - Same: Differentiate $3x^5$.
- If $f(x) = \cos(\pi x + 7)$, then $f'(x) = ?$.
- If $f(x) = \sqrt{5 + xe^x}$, then $f'(x) = ?$.

Monday	Tuesday	Wednesday	Thursday	Friday	Saturday	Sunday
18	19 Lecture	20	21	22	23	24 🔯 Purim
25 🕉 Holi	26 Lecture (today)	27	28	29	30	31 ✝ Easter
1	2	3	4	5	6	7
8	9 Lecture ☪ Eid	10	11	12	13	14

First written assignment due at start of class

The Power Rule

The derivative of x^n is
$n\,x^{n-1}$
if n is any constant.

The Constant Multiple Rule

For any function f and constant c,
$$(c \cdot f(x))' = c \cdot f'(x).$$

The Sum Rule

For any functions f and g,
$$(f(x) + g(x))' = f'(x) + g'(x).$$

Example: Find the derivative of $x^4 - 7x$.

Example: Differentiate $3x^6 + 12\sqrt{x} + 4$.

The derivative of 4 is 0 because

- Power Rule with n=0.
- the slope of a horizontal line is 0.
- the rate of change of f(x) = 4 is 0 (it does't change!)

Calculate the derivative of each of these, if you can:

- x^5
- x^{-3}
- $\frac{7}{x}$
- 9
- $\sqrt{5x}$
- 3^2
- 3^x
- $8x^3$

Calculate the derivative of each of these, if you can:

- x^5 $5x^4$
- x^{-3} $-3x^{-4}$
- $\dfrac{7}{x}$ $-7x^{-2}$
- 9
- $\sqrt{5x}$ $D[\sqrt{5}\, x^{1/2}] = \frac{\sqrt{5}}{2} x^{-1/2}$
- 3^2 $D[3^2] = D[9] = 0$
- 3^x (We haven't learned this one yet.)
- $8x^3$ $24x^2$

The Power Rule does NOT apply here since it's not in the form x^n.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\sin(x)\right] = \cos(x) \qquad \frac{\mathrm{d}}{\mathrm{d}x}\left[\cos(x)\right] = -\sin(x)$$

We still have the Constant Multiple Rule and the Sum Rule, so we can also find derivatives of

- $x^2 + 190 + 2\sin(x)$
- $4x^3 + 6\cos(x) - x$

and similar functions.

We do not *yet* have a rule to find $\frac{\mathrm{d}}{\mathrm{d}x}\left[\sin(2x)\right]$ or $\left(\tan(x)\right)'$.

There are several ways to combine functions:

- SUM: $\sin(x) + \sqrt{x}$.
- DIFFERENCE: $\sin(x) - \sqrt{x}$, and $\sqrt{x} - \sin(x)$.
- PRODUCT: $\sqrt{x} \cdot \sin(x)$.
- QUOTIENT: $\dfrac{\sin(x)}{\sqrt{x}}$, and $\dfrac{\sqrt{x}}{\sin(x)}$.
- COMPOSITION: $\sin\left(\sqrt{x}\right)$, and $\sqrt{\sin(x)}$.

We know how to find derivatives of these already. (SUM, DIFFERENCE)

Now (PRODUCT)

Next week (QUOTIENT, COMPOSITION)

Students on the left:

1. Find $(x^3)'$, meaning the derivative of x^3.
2. Find $(x^2)'$, meaning the derivative of x^2.
3. Simplify $(x^3)' \cdot (x^2)'$.

$(3x^2)(2x) = 6x^3$

↑ not equal ↓

$(x^5)' = 5x^4$

Students on the right:

1. Simplify $x^3 \cdot x^2$.
2. Find the derivative of the function from your step 1.

$(f \cdot g)'$ is NOT $f' \cdot g'$.

Everyone:

1. Find $(x^3)'$.

2. Find $(x^2)'$.

3. Simplify $(x^3) \cdot (x^2)' + (x^3)' \cdot (x^2)$.

Not derivatives!

This does give us exactly $5x^4$, the derivative of $x^3 \cdot x^2$.

Product Rule

$$(f \cdot g)' = f \cdot g' + f' \cdot g$$

We can write the Product Rule with prime notation or fraction notation:

Product Rule

$$(f \cdot g)' = f \cdot g' + f' \cdot g$$

Product Rule

$$\frac{\mathrm{d}}{\mathrm{d}x}[fg] = f\frac{\mathrm{d}g}{\mathrm{d}x} + \frac{\mathrm{d}f}{\mathrm{d}x}g$$

We have just checked that this rule is true for $f(x) = x^2$ and $g(x) = x^3$.

Example: What is the derivative of $x^8 \sin(x)$?

$fg' + f'g = (x^8)(\cos(x)) + (8x^7)(\sin(x))$.

We could also write this as $x^7(x\cos(x) + 8\sin(x))$.

Which of these are products of numbers?

- $5 \cdot 7$ Yes
- $5 \cdot (1 + 2)$ Yes
- $(3 \cdot 2) + 7$ No*

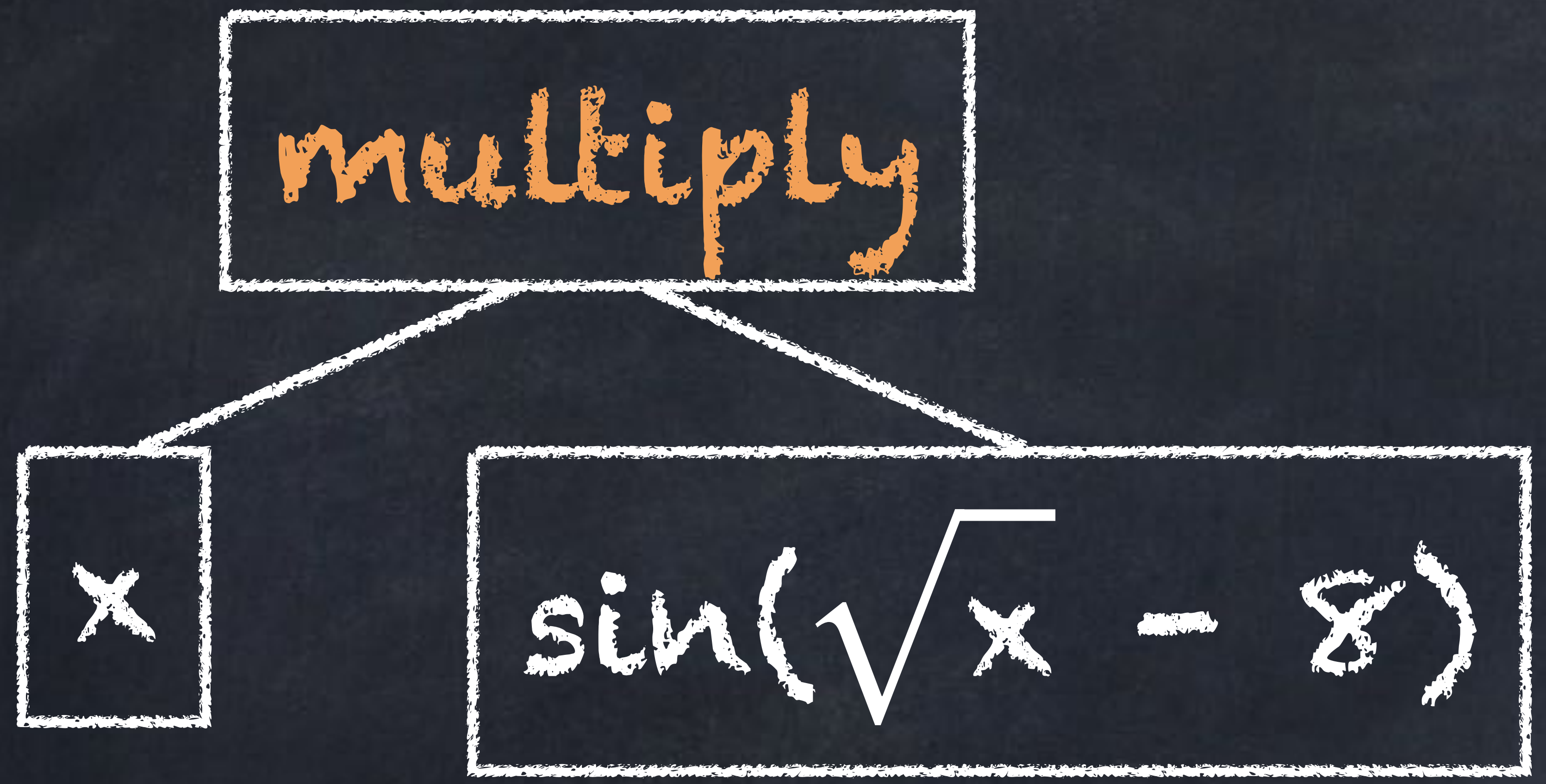
Which of these are products of functions?

- $e^x \sin(x)$ Yes
- $\sqrt{x^7} + x^3$ No*
- $x^3 \ln(x)$
- $x^3\left(x^2 - \cos(x^3)\right)$
- $(3x - 7)(2x + 1)^5$
- $x \ln(\sin(x^3 - 8))$
- $x \sin\left(\sqrt{x} - 8\right)$
- $x \sin\left(\sqrt{x}\right) - 8$ No*

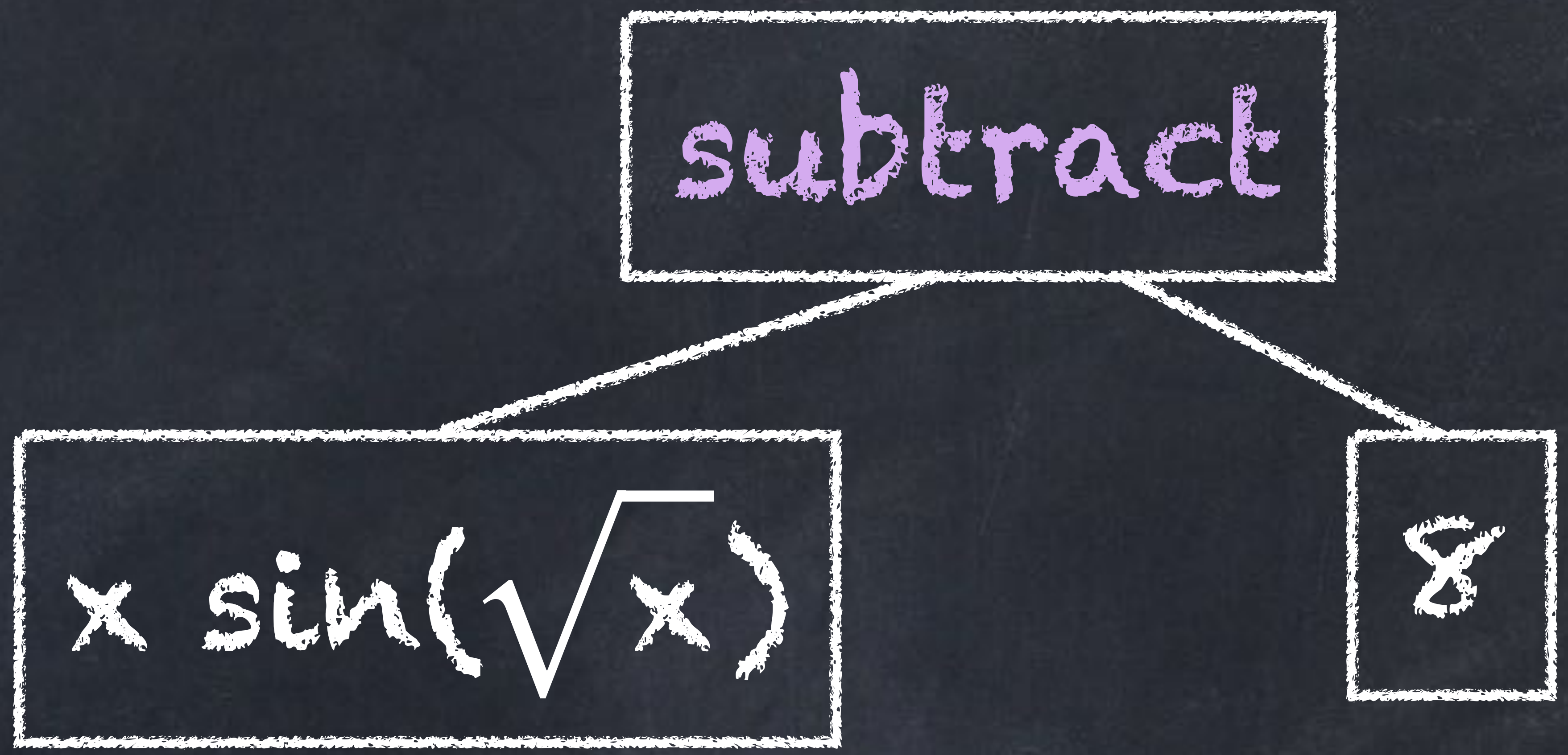
* Technically anything can be a product because you can multiply by 1. The point is that the Product Rule would not be the *first* rule to use for any of the "No" expressions here.

It might help to think of an expression as a “tree”:

$$x \sin\left(\sqrt{x} - 8\right)$$



$$x \sin\left(\sqrt{x}\right) - 8$$



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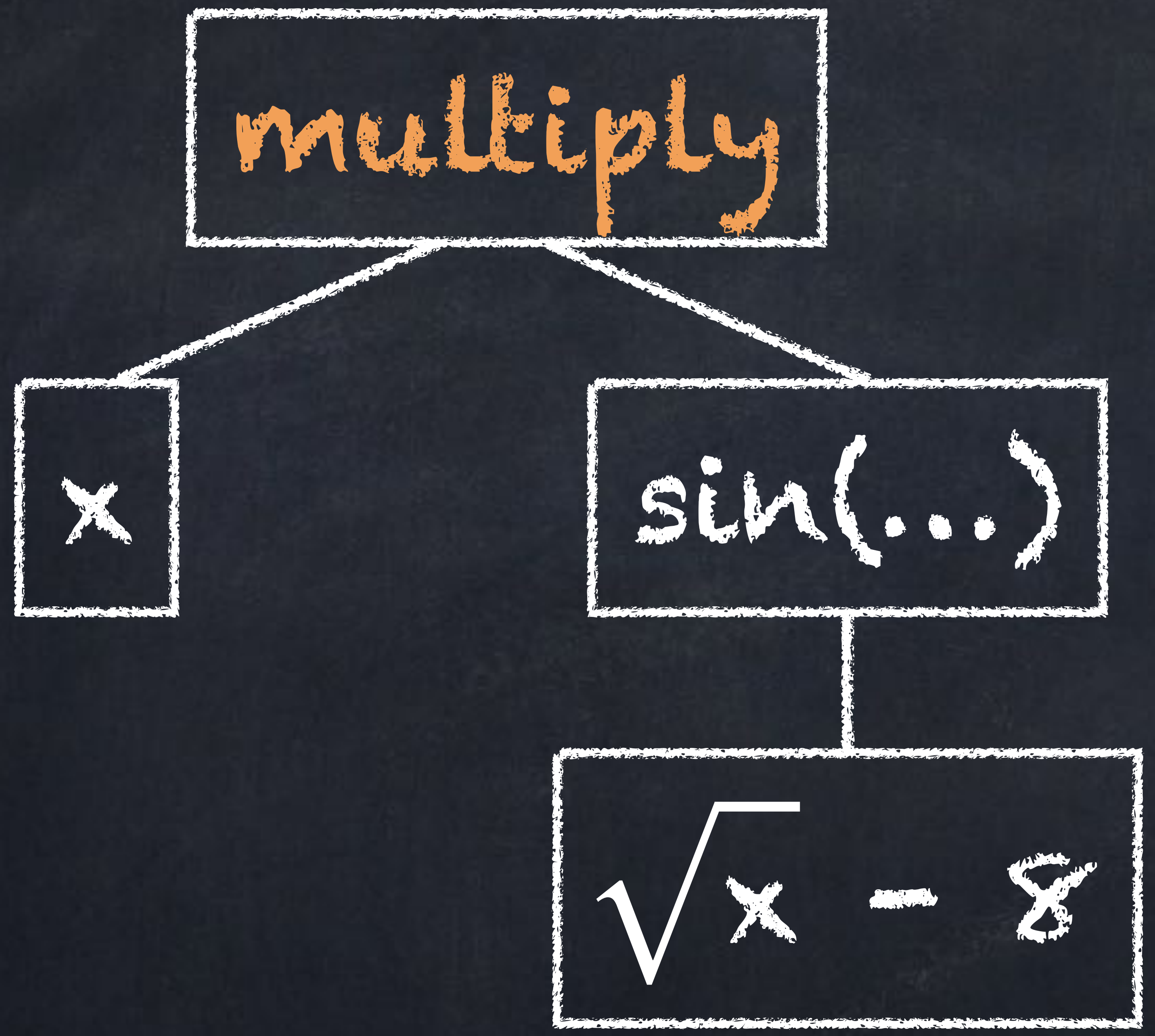
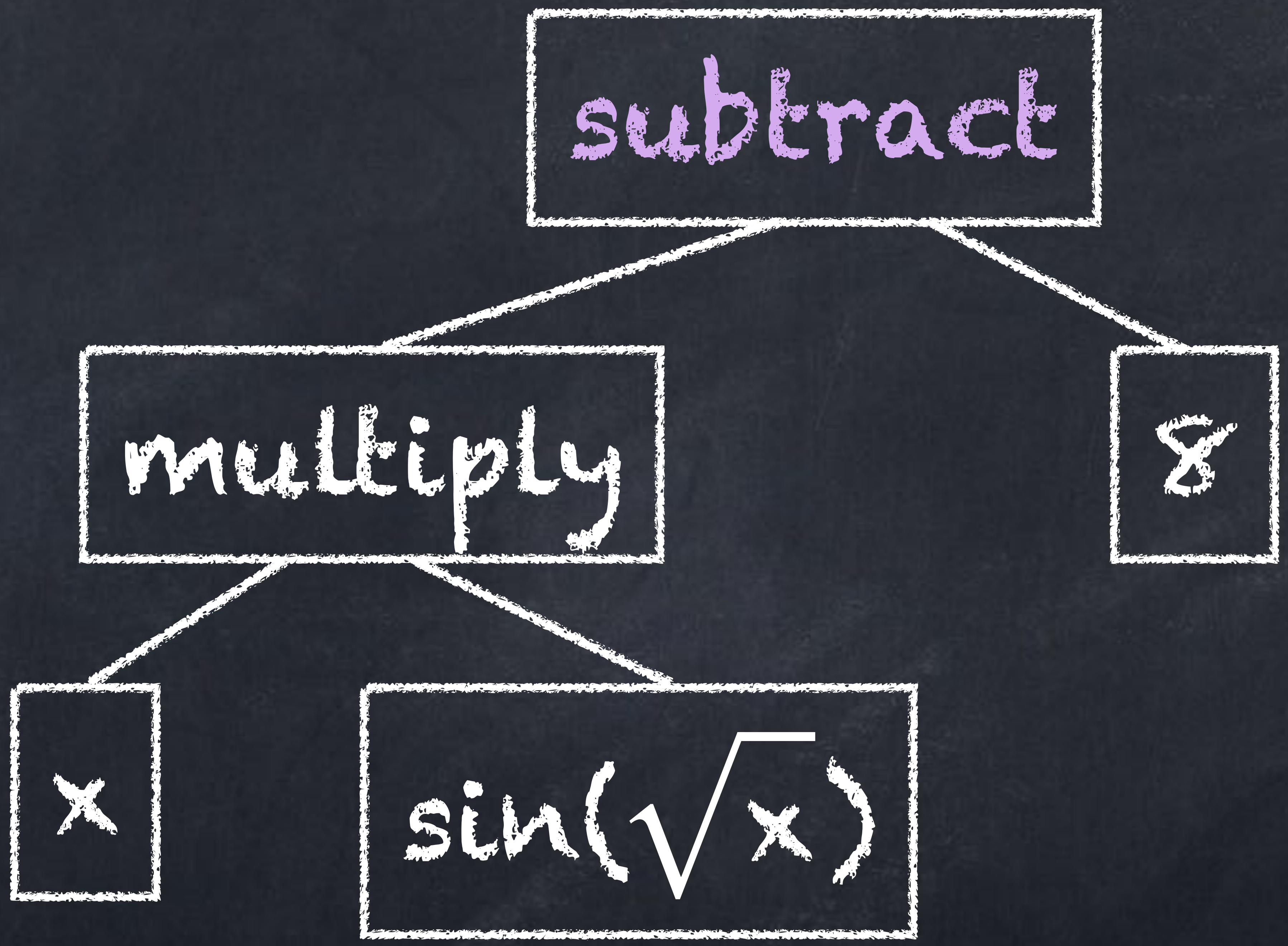
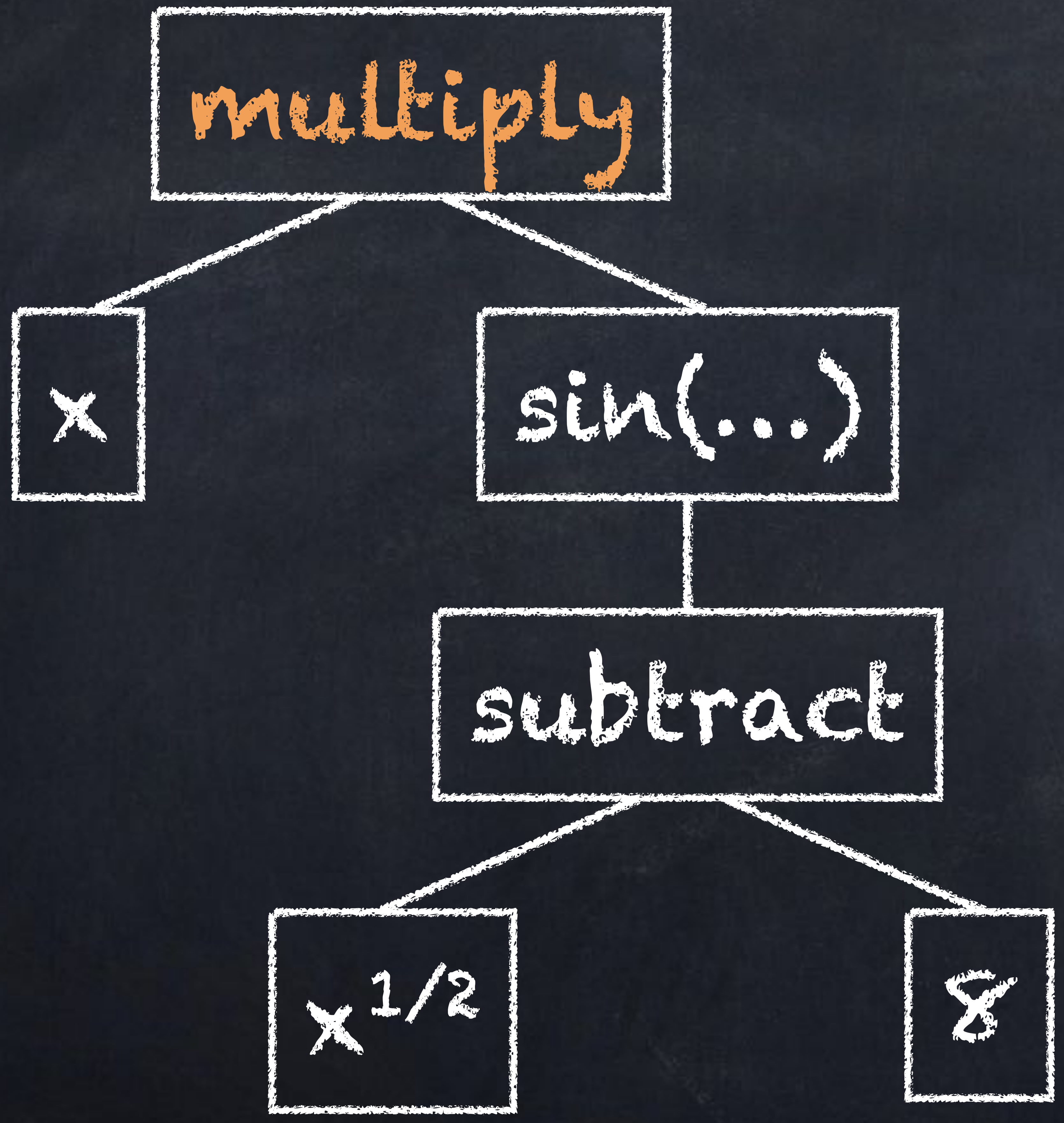


$$x \sin\left(\sqrt{x}\right) - 8$$

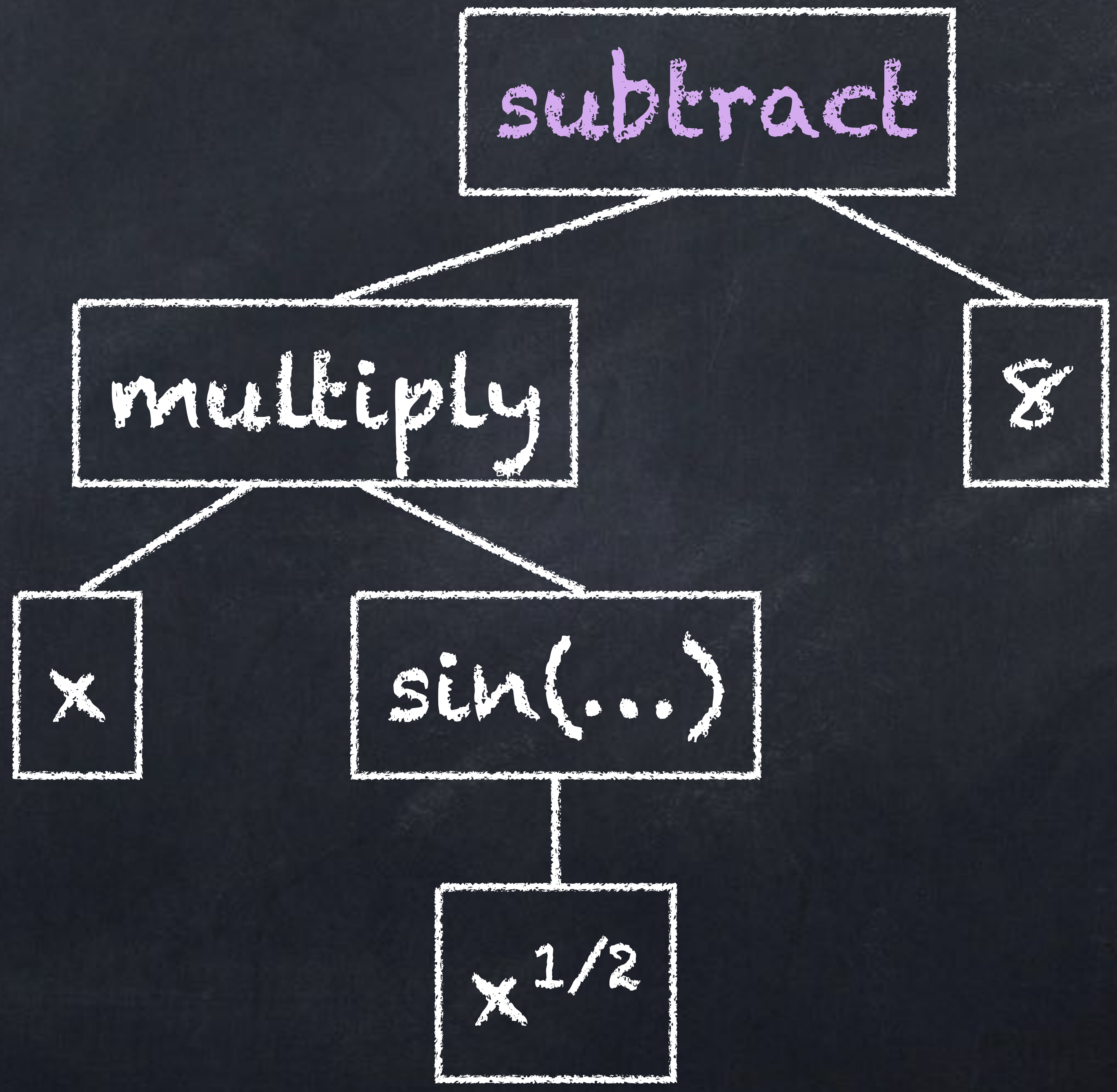


It might help to think of an expression as a "tree":

$$x\sin\left(\sqrt{x}-8\right)$$



$$x\sin\left(\sqrt{x}\right)-8$$



Product Rule

Example: Calculate $\left((2x - 8x^3)\sqrt{x}\right)'$.

Answer: $(2 - 24x^2)\sqrt{x} + \dfrac{1 - 4x^3}{\sqrt{x}}$

Find f' or df/dx.

Differentiate the functions whose letters are the start of your first or last name.

A) $x^2 - 5x + 27$

B) $\frac{1}{2} - x$

C) cx^3

D) $8\sin(x)$

E) $7\cos(x)$

F) $x^2\cos(x)$

G) $6x^{-2}$

H) 1238

I) $\sqrt[3]{x}$

J) $x\cos(x)$

K) $5 - x^3$

L) $(x^2 + 1)(x^{10} - 3)$

Ł) $\frac{2}{\sqrt{x}}$

M) $\frac{-2}{x^5}$

N) $x^{-1/9}$

O) $\sqrt{\sqrt{x}}$

Ö) $7x^2 + 5 + 3x^{-1}$

P) $x^4 - x^3 + x^2 - x + 1$

Q) $5 + \sqrt{5}$

R) $3\sin(x) + 2\cos(x)$

S) $\cos(x) + \sqrt{x}$

T) $\cos(x) \cdot \sqrt{x}$

U) $\cos(x) \cdot \sin(x)$

V) $\sqrt{x^5}$

Y) $6x^{-2} + 5x^2$

Z) x^{100}